

ZEROS OF SECTIONS OF POWER SERIES: DETERMINISTIC AND RANDOM

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ABSTRACT. We present a streamlined proof (and some refinements) of a characterization (due to F. Carlson and G. Bourion, and also to P. Erdős and H. Fried) of the so called Szegő power series. This characterization is then applied to readily obtain some (more) recent known results and some new results on the asymptotic distribution of zeros of sections of random power series, extricating quite naturally the deterministic ingredients. Finally, we study the possible limits of the zero counting probabilities of a power series.

1. INTRODUCTION

The first aim of this paper is to present a streamlined proof and a refined version of a characterization (due to F. Carlson and G. Bourion, and also to P. Erdős and H. Fried) of the so called Szegő power series: theorems 2.6 and 2.7.

That characterization is then applied in Section 5 to readily obtain some (more) recent known results and some new results on the asymptotic distribution of zeros of sections of random power series, extricating quite naturally the deterministic ingredients. Finally, in Section 6 we study the possible limits of the zero counting probabilities associated to a power series.

We shall denote by \mathcal{F} the class of power series whose radius of convergence is 1. The results which we are about to discuss concerning such f can be translated, with obvious scaling, to power series of positive and finite radius of convergence.

For a given power series $f \in \mathcal{F}$ and for each $n \geq 0$, we denote by $s_n = s_n(f)$ the n -th section of the power series: $s_n(z) = \sum_{k=0}^n a_k z^k$, and by \mathcal{Z}_n the (multi-)set of the zeros of s_n . To each non constant s_n we associate two measures: we denote by $\mu_n = \mu_n(f)$ the zero counting measure

$$\mu_n = \frac{1}{n} \sum_{w \in \mathcal{Z}_n} \delta_w,$$

a weighted sum of Dirac deltas placed at the zeros of s_n repeated according to their multiplicity, and we denote by $\rho_n = \rho_n(f)$ the circular projection of μ_n :

$$\rho_n = \frac{1}{n} \sum_{w \in \mathcal{Z}_n} \delta_{|w|}.$$

If $a_n \neq 0$, then μ_n and ρ_n are probability measures. If $a_n = 0$ (and s_n is non constant), we append the definition above by adding a Dirac delta at $\infty_{\mathbb{C}}$ with mass $n - \deg(s_n)$ so that μ_n and ρ_n become probability measures on the Riemann sphere $\hat{\mathbb{C}}$. By F_n we denote the distribution function of ρ_n , given by

$$F_n(t) = \mu_n(|z| \leq t), \quad \text{for } t \geq 0,$$

thus $F_n(t)$ is the average number of zeros of s_n within the disk $\{z \in \mathbb{C} : |z| \leq t\}$.

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By Hurwitz's theorem, $\lim_{n \rightarrow \infty} F_n(t) = 0$ for any $t < 1$; actually, $F_n(t) = O(1/n)$, for any fixed $t < 1$.

We are concerned in this paper with the convergence as $n \rightarrow \infty$ of the probabilities μ_n and ρ_n associated to a given $f \in \mathcal{F}$ and with the potential limits which these probability measures may have. By convergence we mean weak convergence, so that a sequence $(\lambda_n)_{n \geq 0}$ of measures on $\widehat{\mathbb{C}}$ converges to a measure λ_∞ if

$$\lim_{n \rightarrow \infty} \int h d\lambda_n = \int h d\lambda_\infty, \quad \text{for any function } h \text{ bounded and continuous on } \widehat{\mathbb{C}}.$$

We shall denote by Λ the uniform probability (normalized Lebesgue measure) on $\partial\mathbb{D}$. Convergence of the zero counting probabilities μ_n to Λ means that for each h as above,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{w \in \mathcal{Z}_n} h(w) = \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\vartheta}) d\vartheta.$$

In Section 2, we shall discuss the main results about the asymptotic behavior of μ_n and ρ_n for a given $f \in \mathcal{F}$. Section 3 is devoted to results connecting coefficients and zeros of polynomials, and some proofs. Section 4 discusses the aforementioned streamlined proof of the characterization of the Szegő class. This characterization is then applied in Section 5 to the study of the sequences of zero counting measures of random power series. Finally, returning to the deterministic context, Section 6 discusses the possible limits of the zero counting measures of a given f and exhibits an example of a power series whose sequence of ρ_n is dense.

2. ASYMPTOTICS OF ZERO COUNTING MEASURES

Here we describe the main results about asymptotics of zero counting measures, from the seminal work of Jentzsch and Szegő up to the characterization of those holomorphic functions f whose zero counting measures converge to the uniform probability Λ .

Theorem 2.1.

- i) For any $f \in \mathcal{F}$, there is a subsequence $(n_k)_{k \geq 1}$ such that ρ_{n_k} converges to δ_1 .
- ii) Given $f \in \mathcal{F}$, if for a subsequence $(n_k)_{k \geq 1}$ the probability measures ρ_{n_k} converges to δ_1 , then μ_{n_k} converges to Λ , and conversely.

Part i) of Theorem 2.1 is due to Jentzsch, [13]; it claims that there is a subsequence μ_{n_k} of the μ_n asymptotically concentrated on $\partial\mathbb{D}$. Since $\lim_{k \rightarrow \infty} F_{n_k}(t) = 0$ for each $t < 1$, the conclusion of part i) is equivalent to the statement that $\lim_{k \rightarrow \infty} F_{n_k}(T) = 1$, for each $T > 1$.

The second part, ii), of Theorem 2.1, which is due to Szegő, [23], says that just simple radial concentration of the mass of μ_n towards the unit circle $\partial\mathbb{D}$ is equivalent to the (much more precise) statement that the μ_n converge to the uniform probability Λ on $\partial\mathbb{D}$.

The paradigmatic example of Theorem 2.1 is the power series $1/(1-z) = \sum_{k=0}^{\infty} z^k$. In this case \mathcal{Z}_n consists of the $(n+1)$ -th roots of unity except $z = 1$, and the whole sequence μ_n converges to Λ . By contrast, for the lacunary power series $f(z) = \sum_{k=0}^{\infty} z^{2^k}$ a simple application of Rouché's theorem gives that μ_{2^k} converges to Λ , while, since $s_{2^k-1} \equiv s_{2^k-1}$, the probabilities ρ_{2^k-1} converge to $(\delta_1 + \delta_\infty)/2$; the whole sequence of zero counting measures of f does not converge.

For a general treatment and a modern account of the theory of asymptotic distribution of zeros of polynomials we would like to refer to [1]; the reader will find there complete references to the many authors who have contributed to the subject.

We say that a power series $f \in \mathcal{F}$ is a **Szegő power series** (and belongs to the **Szegő class** \mathcal{S}) if the corresponding *complete* sequence of zero counting measures μ_n converges to the measure Λ .

Naturally, we would like to have conditions on the coefficients a_n of the power series f which would imply that f is a Szegő power series. Szegő gave in [23] one first such condition:

Theorem 2.2 (Szegő). *If*

$$(2.1) \quad \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1,$$

then $f \in \mathcal{S}''$.

Condition (2.1) is quite restrictive: for each integer $N \geq 2$, the power series $1/(1 - z^N) = \sum_{k=0}^{\infty} z^{kN}$ belongs to \mathcal{S}'' , but $\liminf_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 0$.

Theorem 2.6 below is Carlson's *characterization* (in terms of the coefficients a_n) of the Szegő class \mathcal{S}'' ; to state it we need to introduce a few concepts and some further notation.

2.1. Gauge and index of power series. Consider a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{F}$. For each $\gamma \in [0, 1)$, define

$$A_n(\gamma) = \max_{(1-\gamma)n \leq k \leq n} |a_k|, \quad \text{for each } n \geq 0,$$

and

$$L(\gamma) = \liminf_{n \rightarrow \infty} \sqrt[n]{A_n(\gamma)}.$$

Observe that $L(\gamma)$ increases with γ and that $0 \leq L(\gamma) \leq 1$, for $\gamma \in [0, 1)$.

We define the **index** Γ of a power series $f \in \mathcal{F}$ as

$$\Gamma := \inf\{\gamma \in (0, 1) : L(\gamma) = 1\}.$$

We set $\Gamma = 1$ if $L(\gamma) < 1$ for each $\gamma \in (0, 1)$; this occurs, for instance, for $\sum_{k=0}^{\infty} z^{k!}$.

The **gauge** G of a power series $f \in \mathcal{F}$ is defined by

$$G := \lim_{\gamma \downarrow 0} L(\gamma) = \inf_{\gamma \in (0, 1)} L(\gamma).$$

Observe that $0 \leq G \leq 1$ and that gauge $G = 1$ is equivalent to index $\Gamma = 0$.

A related but different notion of “gap index” appears in [19], page 277.

Lacunary series like $\sum_{k=0}^{\infty} z^{q^k}$, where q is an integer, $q \geq 2$, have index $\Gamma = 1 - 1/q$ and gauge $G = 0$. More generally,

Lemma 2.3. *For any $t \in (0, 1]$ and any $g \in [0, 1)$, there exists a power series $\sum_{n=0}^{\infty} a_n z^n \in \mathcal{F}$ with index $\Gamma = t$ and gauge $G = g$.*

This lemma, combined with the observation that $G = 1$ is equivalent to $\Gamma = 0$, means that $\{(\Gamma, G) : f \in \mathcal{F}\} = (0, 1] \times [0, 1) \cup \{(0, 1)\}$.

Proof. Let $(m_k)_{k \geq 1}$ be an increasing sequence of positive integers such that $m_k/m_{k+1} \rightarrow 1 - t$, as $k \rightarrow \infty$. Denote by \mathcal{M} the set $\mathcal{M} = \{m_k : k \geq 1\}$.

Define the coefficient sequence $(a_n)_{n \geq 0}$ by $a_n = 1$ if $n \in \mathcal{M}$ and $a_n = g^n$ otherwise. Observe that $\sum_{n=0}^{\infty} a_n z^n \in \mathcal{F}$.

For each $n \geq 0$ and every $\gamma \in (0, 1)$ we have that

$$\sqrt[n]{A_n(\gamma)} = \begin{cases} 1, & \text{if } [n(1-\gamma), n] \cap \mathcal{M} \neq \emptyset, \\ g^{\lceil n(1-\gamma) \rceil / n}, & \text{if } [n(1-\gamma), n] \cap \mathcal{M} = \emptyset. \end{cases}$$

Observe that $L(\gamma) \geq g^{1-\gamma}$, for any $\gamma \in (0, 1)$.

Next we show that

$$L(\gamma) = \begin{cases} 1, & \text{if } 1 > \gamma > t, \\ g^{1-\gamma}, & \text{if } 0 < \gamma < t. \end{cases}$$

Let $1 > \gamma > t$. If $m_k < n \leq m_{k+1}$ then $n(1-\gamma) \leq m_{k+1}(1-\gamma) < m_k$, for k is large enough. Thus $\sqrt[n]{A_n(\gamma)} = 1$, for n large enough, and so $L(\gamma) = 1$.

Let $0 < \gamma < t$. Since $(1-\gamma)(m_{k+1} - 1) > m_k$, for k large enough, we have that $\sqrt[n]{A_n(\gamma)} = g^{\lceil n(1-\gamma) \rceil / n}$, for $n = m_{k+1} - 1$ and k large enough. Therefore, $L(\gamma) \leq g^{1-\gamma}$ and $L(\gamma) = g^{1-\gamma}$.

We conclude that $\Gamma = t$, since $g < 1$, and that $G = \lim_{\gamma \downarrow 0} L(\gamma) = g$, since $t > 0$. \square

Remark 2.4 (Index and Ostrowsky gaps). Power series f with positive index $\Gamma > 0$ are said to have Ostrowsky (or Hadamard–Ostrowsky) gaps. This notion appeared first in Ostrowsky characterization of overconvergent power series: a power series $f \in \mathcal{F}$, analytically continuable beyond the unit circle $\partial\mathbb{D}$, is overconvergent if and only if it has index $\Gamma > 0$.

Remark 2.5 (Gauge and index of rational functions). For $f(z) = 1/(1 - z^N)$, with $N \geq 2$, one has $L(0) = 0$, but $L(\gamma) = 1$ for each $\gamma > 0$, and so f has index $\Gamma = 0$ and gauge $G = 1$.

In general, any power series $f \in \mathcal{F}$ which defines a rational function has index $\Gamma = 0$. For, let $f(z) = \sum_{k=0}^{\infty} a_k z^k = P(z)/Q(z)$, for each $z \in \mathbb{D}$, where P, Q are relatively prime polynomials and let m be the degree of the denominator. Consider

$$\alpha_n = \max\{|a_n|, |a_{n-1}|, \dots, |a_{n-m+1}|\}, \quad \text{for } n \geq m-1.$$

A result of Pólya ([16], Hilfssatz III, and also, [17], problem 243) gives that

$$\lim_{n \rightarrow \infty} \sqrt[n]{\alpha_n} = 1.$$

Note that we have ‘lim’ above, not just ‘lim sup’. For any $\gamma \in (0, 1)$ we have that

$$A_n(\gamma) \geq \alpha_n, \quad \text{for } n \geq m/\gamma;$$

consequently,

$$L(\gamma) = \liminf_{n \rightarrow \infty} \sqrt[n]{A_n(\gamma)} \geq \liminf_{n \rightarrow \infty} \sqrt[n]{\alpha_n} = 1,$$

so that $\Gamma = 0$.

2.2. Characterization of the Szegő class. The following characterization of Szegő power series was announced by F. Carlson, [6]. A complete proof appeared in the monograph [5], page 19, of G. Bourion; later on, P. Erdős and H. Fried, [7], gave an alternative proof of the characterization (and credit Theorem 2.6 to Bourion).

Theorem 2.6 (Carlson-Bourion). *Let f be a power series in \mathcal{F} . Then*

$$f \in \mathcal{S}'' \text{ if and only if its gauge } G \text{ is } 1.$$

We shall derive this theorem from the following:

Theorem 2.7. *Let f be a power series in \mathcal{F} with gauge G . Then*

$$(2.2) \quad \liminf_{n \rightarrow \infty} F_n(T) \geq 1 - \frac{\ln(1/G)}{\ln(T)} > 0, \quad \text{for each } T > 1/G.$$

And also,

$$(2.3) \quad \liminf_{n \rightarrow \infty} F_n(T) < 1, \quad \text{for each } T < 1/G.$$

Recall, in addition, that for any $t < 1$ we always have that $\lim_{n \rightarrow \infty} F_n(t) = 0$.

Theorem 2.6 may be derived from Theorem 2.7 as follows:

a) If $G = 1$, equation (2.2) tells that $\lim_{n \rightarrow \infty} F_n(T) = 1$ for each $T > 1$ and then Theorem 2.1, ii) says that μ_n converges to Λ , and, so, $f \in \mathcal{S}''$.

b) Equation (2.3) combined with a diagonal argument implies that if $G < 1$, a subsequence of the ρ_n converges to a probability measure whose essential support reaches $1/G$.

In particular, the ρ_n does not converge to δ_1 and $f \notin \mathcal{S}''$.

We shall give a proof of Theorem 2.7 in Section 4.

3. COEFFICIENTS AND ZEROS OF POLYNOMIALS

Next we collect a number of general results connecting coefficients and zeros of polynomials. We also give proofs, based on those connections and to be used later on, of Theorems 2.1 and 2.2.

For a polynomial P and integer $n \geq \deg(P)$, we write P as $P(z) = \sum_{k=0}^n b_k z^k$, where $b_k = 0$ for $\deg(P) < k \leq n$. We let $\mathcal{Z}_{(P,n)}$ denote its zero (multi-)set, maintaining the convention that if $\deg(P) < n$, then P has a zero of multiplicity $n - \deg(P)$ at $\infty_{\mathbb{C}}$.

For each $t \geq 0$, we denote by $F_{(P,n)}(t)$ the proportion (with respect to n) of the zeros of P within the disk $\{w \in \mathbb{C} : |w| \leq t\}$. Notice that $\lim_{t \rightarrow \infty} F_{(P,n)}(t) = \deg(P)/n$.

We shall frequently appeal to the *reversed companion polynomial* Q of P with respect to n :

$$Q_{(P,n)}(z) = z^n P(1/z) = \sum_{k=0}^n b_{n-k} z^k;$$

the zeros of $Q_{(P,n)}$ are the reciprocals of the zeros of P .

Furthermore, we order the zeros of P according to their modulus and denote them by w_1, w_2, \dots, w_n :

$$|w_1| \leq |w_2| \leq \dots \leq |w_n|,$$

keeping in mind that the last $n - \deg(P)$ of those are $= \infty_{\mathbb{C}}$.

3.1. Jensen's formula. For $n \geq \deg(P)$, an application of Jensen's formula to both P and its reversed companion $Q_{(P,n)}$ inside the unit disk gives that

$$\sum_{w \in \mathcal{Z}_{(P,n)}} |\ln |w|| = \frac{1}{2\pi} \int_0^{2\pi} \ln \frac{|P(e^{i\vartheta})|^2}{|b_0||b_n|} d\vartheta.$$

Now, for any $T > 1$ we have

$$\begin{aligned} \frac{1}{n} \sum_{w \in \mathcal{Z}_{(P,n)}} |\ln |w|| &\geq \frac{1}{n} \sum_{\substack{w \in \mathcal{Z}_{(P,n)}; \\ |w| > T}} |\ln |w|| + \frac{1}{n} \sum_{\substack{w \in \mathcal{Z}_{(P,n)}; \\ |w| < 1/T}} |\ln |w|| \\ &\geq (\ln T)(1 - F_{(P,n)}(T)) + (\ln T)F_{(P,n)}(1/T), \end{aligned}$$

and, therefore,

$$(3.1) \quad \ln(T) (1 - F_{(P,n)}(T) + F_{(P,n)}(1/T)) \leq \frac{1}{2\pi} \int_0^{2\pi} \ln \frac{\sqrt[n]{|P(e^{i\vartheta})|^2}}{\sqrt[n]{|b_0|} \sqrt[n]{|b_n|}} d\vartheta, \quad \text{for all } T > 1.$$

This inequality (3.1) readily gives a **proof of Theorem 2.1 i) and of Theorem 2.2**. Consider $f \in \mathcal{F}$. Assume without loss of generality that $a_0 = 1$. Apply (3.1) to the partial sum s_n to get

$$(3.2) \quad \ln(T) (1 - F_n(T) + F_n(1/T)) \leq \frac{1}{2\pi} \int_0^{2\pi} \ln \frac{\sqrt[n]{|s_n(e^{i\vartheta})|^2}}{\sqrt[n]{|a_n|}} d\vartheta, \quad \text{for all } T > 1.$$

Since the radius of convergence of f is 1, one has that

$$\limsup_{n \rightarrow \infty} \max_{|z| \leq R} \sqrt[n]{|s_n(z)|} = R, \quad \text{for any } R \geq 1.$$

Let $(n_k)_{k \geq 1}$ be any increasing sequence such that $\lim_{k \rightarrow \infty} |a_{n_k}|^{1/n_k} = 1$. Since, for each $T > 1$, one has $\lim_{n \rightarrow \infty} F_n(1/T) = 0$, we obtain from (3.2) that

$$\ln(T) \limsup_{k \rightarrow \infty} (1 - F_{n_k}(T)) \leq 0, \quad \text{for any } T > 1,$$

and so $\lim_{k \rightarrow \infty} F_{n_k}(T) = 1$, for any $T > 1$. This proves both Theorem 2.1 i), and also Theorem 2.2.

Notice that actually the argument gives the following general inequality:

$$(3.3) \quad \ln(T) (1 - \liminf_{n \rightarrow \infty} F_n(T)) \leq -\ln (\liminf_{n \rightarrow \infty} \sqrt[n]{|a_n|}), \quad \text{for any } T > 1.$$

Cf. [8] and [14].

3.2. Coefficients as symmetric functions of the zeros. For a polynomial P and integer $n \geq \deg(P)$, if $b_0 \neq 0$, the product of the zeros of P and the coefficients of P are related by

$$(3.4) \quad \frac{|b_0|}{|b_n|} = \prod_{w \in \mathcal{Z}_{(P,n)}} |w|.$$

This identity readily gives (another) **proof of Theorem 2.2**. For $f \in \mathcal{F}$, assume without loss of generality that $f(0) = 1$. Fix $t < 1$. From Hurwitz's theorem and the fact that $f(0) \neq 0$, we obtain a constant $K_t > 0$ (which depends on t but not on n) such that

$$\prod_{w \in \mathcal{Z}_n: |w| \leq t} |w| \geq K_t.$$

For each $T > 1$, after classifying the roots w as $|w| \leq t$, $t < |w| \leq T$, and $|w| > T$, we may bound

$$\frac{1}{|a_n|} \geq K_t t^{n(F_n(T) - F_n(t))} T^{n(1 - F_n(T))} \geq K_t t^n T^{n(1 - F_n(T))}.$$

Taking n -th roots and then limits as $n \rightarrow \infty$, we conclude that if $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$ then

$$T^{\liminf_{n \rightarrow \infty} F_n(T)} \geq tT.$$

Since this is valid for any $t < 1$ and since $T > 1$, we deduce that $\liminf_{n \rightarrow \infty} F_n(T) \geq 1$ and thus that $\lim_{n \rightarrow \infty} F_n(T) = 1$. This is Szegő's own argument in [23] to prove Theorem 2.2.

As we shall see, equation (2.2) of Theorem 2.7 will follow from a variation of Szegő's argument but involving more coefficients and not just a_n and equation (3.4).

From the expression of the coefficients of a polynomial P as symmetric functions of its zeros (Viète's formulas) one obtains, for $n \geq \deg(P)$, the inequality

$$(3.5) \quad \frac{|b_k|}{|b_n|} \leq \binom{n}{k} \prod_{j=k+1}^n |w_j|, \quad 0 \leq k \leq n,$$

with the convention that an empty product is 1. Upon considering the reversed companion polynomial $Q_{(P,n)}(z)$, one obtains the inequality

$$(3.6) \quad \prod_{j=1}^k |w_j| \leq \binom{n}{k} \frac{|b_0|}{|b_k|}, \quad 0 \leq k \leq n.$$

To control the binomial coefficients appearing in (3.6) we shall use the known elementary bound

$$(3.7) \quad \binom{n}{k} \leq e^{nH(k/n)}, \quad \text{for } 0 \leq k \leq n \text{ and } n \geq 1,$$

where H denotes the entropy function: $H(x) = -(x \ln(x) + (1-x) \ln(1-x))$ for $x \in [0, 1]$. Notice that $H(x) = 0$ if $x = 0$ or $x = 1$, and that H decreases as x goes from $1/2$ to 1 .

3.3. A proof of Szegő's Theorem 2.1, ii). What follows is a slight simplification of Szegő's own argument for Theorem 2.1, ii).

We assume with no loss of generality that $a_0 = 1$. Since $f(0) = a_0 \neq 0$, we may fix $r > 0$ and integer $N \geq 1$ such that no root of s_n lies in the disk $\{|z| < r\}$.

For $z \in \mathbb{C} \setminus \{0\}$, we write $z/|z| = e^{i\theta(z)}$ with $\theta(z) \in [0, 2\pi)$.

We shall prove that

$$(3.8) \quad \lim_{k \rightarrow \infty} \frac{1}{n_k} \sum_{w \in \tilde{\mathcal{Z}}_{n_k}} e^{-im\theta(w)} = 0, \quad \text{for any integer } m \geq 1,$$

where $\tilde{\mathcal{Z}}_n$ means \mathcal{Z}_n with $\infty_{\mathbb{C}}$ excluded. Since $\rho_{n_k} \rightarrow \delta_1$ as $k \rightarrow \infty$, the conclusion of Theorem 2.1, ii), will follow from (3.8) combined with Weierstrass approximation theorem.

To prove (3.8), fix an integer $m \geq 1$.

Let σ_n denote the reversed companion polynomial $\sigma_n(z) = \sum_{k=0}^n a_{n-k} z^k$ of the partial sum s_n with respect to n . An application of Newton's identities to σ_n gives that

$$\sum_{w \in \mathcal{Z}_n} \frac{1}{w^m} = \sum_{w \in \tilde{\mathcal{Z}}_n} \frac{1}{w^m} = \Psi_m(a_1, a_2, \dots, a_m),$$

where Ψ_m is a certain function defined in \mathbb{C}^m . Consequently,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{w \in \mathcal{Z}_n} \frac{1}{w^m} = 0.$$

Now, for $n \geq N$, write

$$\frac{1}{n} \sum_{w \in \tilde{\mathcal{Z}}_n} e^{-im\theta(w)} = \frac{1}{n} \sum_{w \in \mathcal{Z}_n} \frac{1}{w^m} + \frac{1}{n} \sum_{w \in \tilde{\mathcal{Z}}_n} e^{-im\theta(w)} (1 - |w|^{-m}).$$

For $T > 1$, we may bound the last sum in the expression above as

$$\begin{aligned} \left| \frac{1}{n} \sum_{w \in \tilde{\mathcal{Z}}_n} e^{-im\theta(w)} (1 - |w|^{-m}) \right| &\leq \frac{1}{n} \sum_{w \in \tilde{\mathcal{Z}}_n} |1 - |w|^{-m}| \\ &\leq F_n(1/T)(r^{-m} - 1) + (T^m - 1) \\ &\quad + (1 - F_n(T)) + (1 - T^{-m}). \end{aligned}$$

Since $\lim_{n \rightarrow \infty} F_n(1/T) = 0$ and, by hypothesis, $\lim_{k \rightarrow \infty} (1 - F_{n_k}(T)) = 0$, we conclude that

$$\limsup_{k \rightarrow \infty} \left| \frac{1}{n_k} \sum_{w \in \tilde{\mathcal{Z}}_{n_k}} e^{-im\theta(w)} \right| \leq T^m - T^{-m}, \quad \text{for any } T > 1,$$

which gives (3.8).

3.4. Cauchy's and Van Vleck's bounds. These are classical bounds for the location of zeros of a polynomial P in terms of (all or some of) its coefficients. Consult [15], chapters VII and VIII, or [9], chapter 6, and also the original paper [24] of Van Vleck.

The bound of Cauchy asserts that all the zeros of the polynomial P lie in $\{w \in \mathbb{C} : |w| \leq C_P\}$, where C_P is the unique positive root of

$$|b_n| x^n = \sum_{k=0}^{n-1} |b_k| x^k.$$

We understand that if $b_n = 0$, then $C_P = +\infty$.

Upon considering the reversed companion polynomial $Q_{(P,n)}$ one observes that all the roots of P lie in $\{w \in \mathbb{C} : |w| \geq c_P\}$ where c_P is the unique positive root of the equation in y :

$$|b_0| = \sum_{k=1}^n |b_k| y^k.$$

Notice that

$$|b_0| \leq n \left(\max_{1 \leq k \leq n} |b_k| \right) \max(1, c_P)^n.$$

Van Vleck's bounds assert that for $1 \leq m \leq n$ at least m zeros of P lie in the disk $\{w : |w| \leq V_P^m\}$, where V_P^m is the unique positive root of

$$|b_n| x^n = \sum_{j=0}^{m-1} \binom{n-j-1}{m-j-1} |b_j| x^j.$$

The case $m = n$ is Cauchy's bound: $V_P^n = C_P$. Again, we understand that $V_P^m = +\infty$ if $b_n = 0$.

Upon applying these bounds to the reversed companion polynomial $Q_{(P,n)}$ with respect to n we deduce that, if $1 \leq m \leq n$, the polynomial P has m roots in $\{w \in \mathbb{C} : |w| \geq v_P^m\}$, where v_P^m is the unique positive root of the equation

$$|b_0| = \sum_{k=n-m+1}^n \binom{k-1}{k-(n-m)-1} |b_k| y^k.$$

Using that

$$\sum_{k=n-m+1}^n \binom{k-1}{k-(n-m)-1} = \binom{n}{m-1},$$

we deduce that

$$(3.9) \quad |b_0| \leq \binom{n}{m-1} \left(\max_{n-m+1 \leq k \leq n} |b_k| \right) \max(1, v_P^m)^n.$$

4. PROOF OF THEOREM 2.7

Let $f \in \mathcal{F}$. We assume with no loss of generality that $a_0 = 1$.

First we deal with the **proof of inequality** (2.2).

For each $n \geq 1$, present the zeros of s_n in ascending order of modulus as $|w_1^{(n)}| \leq \dots \leq |w_n^{(n)}|$. Recall that if the degree of s_n is $m \leq n$, we, conveniently and conventionally, understand that the last $n - m$ of these zeros are $\infty_{\mathbb{C}}$.

The bounds (3.6) translate into

$$\prod_{j=1}^k |w_j^{(n)}| \leq \binom{n}{k} \frac{1}{|a_k|}, \quad \text{for } 0 \leq k \leq n \text{ and } n \geq 1.$$

Fix $T > 1$ and $\gamma \leq 1/2$. Fix also $t < 1$, which later on will tend to 1. From Szegő's argument of Section 3, maintaining the notation therein, we obtain that

$$K_t t^n T^{k-nF_n(T)} \leq \binom{n}{k} \frac{1}{|a_k|}, \quad \text{for } 0 \leq k \leq n \text{ and } n \geq 1.$$

If we restrict k to the range $(1-\gamma)n \leq k \leq n$ we deduce, using the bound (3.7), that

$$K_t t^n T^{n(1-\gamma-F_n(T))} \leq e^{nH(1-\gamma)} \frac{1}{|a_k|}, \quad \text{for } (1-\gamma)n \leq k \leq n \text{ and } n \geq 1.$$

and, then, that

$$K_t t^n T^{n(1-\gamma-F_n(T))} \leq e^{nH(1-\gamma)} \frac{1}{A_n(\gamma)}, \quad n \geq 1.$$

Upon extracting n -th roots, letting first $n \rightarrow \infty$, and then letting $t \uparrow 1$, we deduce

$$T^{(1-\gamma-\liminf_{n \rightarrow \infty} F_n(T))} \leq e^{H(1-\gamma)} \frac{1}{L(\gamma)}.$$

Letting $\gamma \downarrow 0$, and using that $H(1) = 0$, we deduce that

$$T^{1-\liminf_{n \rightarrow \infty} F_n(T)} \leq \frac{1}{G},$$

or, as claimed,

$$\liminf_{n \rightarrow \infty} F_n(T) \geq 1 - \frac{\ln(1/G)}{\ln(T)}.$$

(Compare this last inequality with inequality (3.3).)

Next, we turn to the **verification of inequality** (2.3).

We assume that $G < 1$, since otherwise the result is trivially true, and we let $1 < T < 1/G$. Fix $\varepsilon > 0$ so that $(G + \varepsilon)T < 1$. Since $H(0) = 0$, we may choose, and fix, $\gamma \in (0, 1/2)$ so that

$$L(\gamma) e^{H(\gamma)} \leq G + \varepsilon/2.$$

For an infinite subset \mathcal{N} of \mathbb{N} one has that

$$(4.1) \quad \sqrt[n]{A_n(\gamma)} e^{H(\gamma)} \leq G + \varepsilon, \quad \text{for } n \in \mathcal{N}.$$

For $n \in \mathcal{N}$, let $m_n = \lfloor \gamma n \rfloor + 1$. The Van Vleck's bounds, equation (3.9), applied to s_n gives that s_n has at least m_n roots with modulus no less than v_n , where v_n satisfies

$$(4.2) \quad 1 \leq \binom{n}{\lfloor \gamma n \rfloor} A_n(\gamma) \max(1, v_n)^n, \quad \text{for } n \in \mathcal{N}.$$

Observe that

$$F_n(v_n) \leq \frac{n - m_n}{n} \leq 1 - \gamma, \quad \text{for } n \in \mathcal{N}.$$

From the bound (3.7) and inequality (4.1) above, we deduce from inequality (4.2) that

$$1 \leq (G + \varepsilon) \max(1, v_n), \quad \text{for } n \in \mathcal{N}.$$

Since $G + \varepsilon < 1$, this means that $v_n > 1$ and, in fact, that

$$\frac{1}{G + \varepsilon} < v_n, \quad \text{for } n \in \mathcal{N}.$$

Therefore

$$F_n(T) \leq 1 - \gamma, \quad \text{for } n \in \mathcal{N},$$

and consequently

$$\liminf_{n \rightarrow \infty} F_n(T) \leq 1 - \gamma < 1.$$

This completes the proof of Theorem 2.7. \square

It should be mentioned that the proof above of equation (2.2) of Theorem 2.7 is a direct adaptation of Szegő's own argument in [23] to prove his Theorem 2.2, which we have discussed in Section 3, while the proof of (2.3) of Theorem 2.7 is a refinement of a suggestion of P. Turán which appears as a *note added in proof* in the paper [7] of Erdős and Fried.

Remark 4.1. The proof above of Theorem 2.7 actually gives that if $\gamma \in (0, 1)$ then

$$\liminf_{n \rightarrow \infty} F_n(T) \leq 1 - \gamma, \quad \text{for any } T < (L(\gamma)e^{H(\gamma)})^{-1}.$$

In the argument above we have just used the case γ close to 0, but if we let $\gamma \uparrow 1$ we obtain:

$$\liminf_{n \rightarrow \infty} F_n(T) = 0, \quad \text{for any } T < (\lim_{\gamma \uparrow 1} L(\gamma))^{-1}.$$

Of course, this is only relevant if $\lim_{\gamma \uparrow 1} L(\gamma) < 1$. Power series for which this occurs, like $\sum_{n=0}^{\infty} z^{n!}$, are said to have *infinite* Ostrowsky gaps, see [7] Theorem II.

5. RANDOM POWER SERIES

Next we turn to random power series. Our aim is to analyze, using the deterministic machinery of the previous sections, the probability that such a random power series is a Szegő power series. We point out to [3] and [10] as general references on random polynomials and on random power series.

5.1. The iid case. To start with, let X be any non null complex valued random variable. Consider a sequence $(X_n)_{n \geq 0}$ of completely independent clones of X in a certain probability space (Ω, \mathbf{P}) . For each $\omega \in \Omega$, let f_ω denote the power series

$$f_\omega(z) = \sum_{k=0}^{\infty} X_k(\omega) z^k.$$

This model of random power series is usually called *Kac ensemble*, particularly so if X is a gaussian variable.

The radius of convergence of f_ω is a random variable, but it turns out to be almost surely constant; actually, the Borel–Cantelli lemma gives directly the following well-known dichotomy.

Lemma 5.1.

If $\mathbf{E}(\ln^+ |X|) < +\infty$, then $\limsup_{n \rightarrow \infty} \sqrt[n]{|X_n|} = 1$, almost surely.

If $\mathbf{E}(\ln^+ |X|) = +\infty$, then $\limsup_{n \rightarrow \infty} \sqrt[n]{|X_n|} = +\infty$, almost surely.

In terms of the power series f_ω , this lemma means that if $\mathbf{E}(\ln^+ |X|) < +\infty$, then the radius of convergence of f_ω is almost surely 1, while if $\mathbf{E}(\ln^+ |X|) = \infty$, the radius of convergence of f_ω is almost surely 0. In other terms, under the condition $\mathbf{E}(\ln^+ |X|) < +\infty$, the random power series f_ω is almost surely in \mathcal{F} .

Notice that if $\mathbf{E}(|\ln |X||) < +\infty$ then almost surely $\lim_{n \rightarrow \infty} \sqrt[n]{|X_n|} = 1$, and conversely. Thus, if $\mathbf{E}(|\ln |X||) < +\infty$, condition (2.1) of Theorem 2.2 holds almost surely.

We include a proof of lemma 5.1, since later on we shall adapt it to the case of non identically distributed random coefficients.

Proof. If $\mathbf{E}(\ln^+ |X|) < +\infty$ then $\sum_{n=0}^{\infty} \mathbf{P}(|X| \geq e^{\alpha n}) < +\infty$, for all $\alpha > 0$. Since the X_n are identically distributed this, in turn, is equivalent to $\sum_{n=0}^{\infty} \mathbf{P}(|X_n| \geq e^{\alpha n}) < +\infty$. The lemma of Borel–Cantelli (no independence needed) gives then, for each $\alpha > 0$, that $\limsup_{n \rightarrow \infty} \sqrt[n]{|X_n|} \leq e^\alpha$ almost surely, and consequently, $\limsup_{n \rightarrow \infty} \sqrt[n]{|X_n|} \leq 1$ almost surely.

Since X is non null, for some $\delta > 0$ we have that $\mathbf{P}(|X| \geq \delta) > 0$. Since the X_n are identically distributed this implies that $\sum_{n=0}^{\infty} \mathbf{P}(|X_n| \geq \delta) = +\infty$. Now, using independence, the lemma of Borel–Cantelli gives then that $\limsup_{n \rightarrow \infty} \sqrt[n]{|X_n|} \geq 1$, almost surely.

If $\mathbf{E}(\ln^+ |X|) = +\infty$, then $\sum_{n=0}^{\infty} \mathbf{P}(|X| \geq e^{\alpha n}) = +\infty$, for all $\alpha > 0$. Now, independence and the lemma of Borel–Cantelli gives that $\limsup_{n \rightarrow \infty} \sqrt[n]{|X_n|} \geq e^\alpha$ for all $\alpha > 0$, and so $\limsup_{n \rightarrow \infty} \sqrt[n]{|X_n|} = +\infty$ almost surely. \square

Fix a non null random variable X and, as above, let $(X_n)_{n \geq 0}$ be a sequence of completely independent clones of X . For $\gamma \in (0, 1)$ and $n \geq 0$, define the random variable

$$A_n(\gamma) = \max_{(1-\gamma)n \leq k \leq n} |X_k|.$$

Lemma 5.2. *If X is a non null random variable, then for each $\gamma \in (0, 1)$*

$$\liminf_{n \rightarrow \infty} \sqrt[n]{A_n(\gamma)} \geq 1 \quad \text{almost surely.}$$

Notice that in Lemma 5.2 no assumption on $\mathbf{E}(\ln^+ |X|)$ is required; just the trivial assumption that X is non null (and independence of the clones, of course) implies that almost surely the sequence $(X_n(\omega))_{n \geq 1}$ can not be too small for long stretches of n .

Observe also that Lemma 5.2 does not hold for $\gamma = 0$: simply take X to be a Bernoulli random variable.

Proof. Fix $\gamma \in (0, 1)$. We have to verify that

$$\mathbf{P}\left(\liminf_{n \rightarrow \infty} \sqrt[n]{A_n(\gamma)} \geq 1\right) = 1,$$

or, equivalently, that, for each $\varepsilon > 0$:

$$\mathbf{P}\left(\sqrt[n]{A_n(\gamma)} \leq (1 - \varepsilon), \text{ infinitely many } n\right) = 0,$$

which, in turn, by the lemma of Borel–Cantelli (no independence assumption needed) reduces to prove that

$$\sum_{n=0}^{\infty} \mathbf{P}\left(\sqrt[n]{A_n(\gamma)} \leq (1 - \varepsilon)\right) < +\infty.$$

Since each X_n is a clone of X , all we have to show is that

$$\sum_{n=0}^{\infty} \mathbf{P}(|X| \leq (1 - \varepsilon)^n)^{\gamma n} < +\infty.$$

Since X is non null, there is $\delta > 0$ such that $\mathbf{P}(|X| < \delta) < 1$. Now for $n \geq N = N(\delta, \varepsilon)$ one has that $(1 - \varepsilon)^n < \delta$, and, consequently,

$$\sum_{n \geq N}^{\infty} \mathbf{P}(|X| \leq (1 - \varepsilon)^n)^{\gamma^n} \leq \sum_{n \geq N}^{\infty} \mathbf{P}(|X| < \delta)^{\gamma^n} < +\infty. \quad \square$$

Theorem 5.3. *If X is a (non-null) random variable and $\mathbf{E}(\ln^+ |X|) < +\infty$, then almost surely the gauge G of f_ω is 1.*

Proof. The assumption $\mathbf{E}(\ln^+ |X|) < +\infty$, implies, by lemma 5.1, that almost surely the radius of convergence of f_ω is 1. And then, the hypothesis that X is non null implies, by Lemma 5.2, that almost surely f_ω has gauge 1. \square

As a consequence of Theorem 5.3 and Theorem 2.6 we obtain the following theorem of Ibragimov and Zaporozhets, [11]. Consult also [2], [14] and [22].

Theorem 5.4 (Ibragimov–Zaporozhets). *For any (non null) random variable X satisfying $\mathbf{E}(\ln^+ |X|) < +\infty$, the sequence $(\mu_n)_{n \geq 0}$ of random probabilities converges almost surely to the uniform probability Λ on $\partial\mathbb{D}$; in other terms, almost all power series f_ω are Szegő power series.*

Remark 5.5. Under the (stronger) hypothesis $\mathbf{E}(|\ln |X||) < +\infty$, Szegő's condition (2.1) is almost surely satisfied and, in this case, one obtains the conclusion of Theorem 5.4 directly from Theorem 2.2, and there is no need to appeal to Theorem 2.6. See also [2].

Remark 5.6. If X is a Bernoulli variable with $\mathbf{P}(X = 1) = p \in (0, 1)$, then almost all f_ω belong to \mathcal{S} , but almost none of the f_ω satisfy the condition (2.1) of Theorem 2.2.

Expected distribution function in the iid case. For each $\omega \in \Omega$, denote by $\mu_{n,\omega}$ and $\rho_{n,\omega}$ the probability measures associated to f_ω and let $F_{n,\omega}(t), t \geq 0$, denote the distribution function of $\rho_{n,\omega}$.

Consider the expected distribution function

$$\Phi_n(t) = \int_{\Omega} F_{n,\omega}(t) d\mathbf{P}(\omega), \quad \text{for } t \geq 0.$$

Since the X_j are completely independent and identically distributed, the section $s_{n,\omega}(z) = \sum_{k=0}^n X_k(\omega) z^k$ and its reversed companion $\sum_{k=0}^n X_{n-k}(\omega) z^k$ are identically distributed and, consequently, the following symmetry holds:

$$\Phi_n(t) = 1 - \Phi_n(1/t), \quad \text{for any } 0 < t \leq 1.$$

Notice that $\Phi_n(1) = 1/2$, for each $n \geq 1$.

Recall that, by Hurwitz's theorem, $\lim_{n \rightarrow \infty} F_{n,\omega}(t) = 0$, for each $t < 1$, almost surely, and so, by dominated convergence, $\lim_{n \rightarrow \infty} \Phi_n(t) = 0$ for each $t < 1$. Consequently, $\lim_{n \rightarrow \infty} \Phi_n(T) = 1$, for each $T > 1$. The last convergence statement follows also from the fact the $G = 1$ almost surely and from Theorems 2.6 and 2.7. Therefore,

$$\lim_{n \rightarrow \infty} \Phi_n(t) = \begin{cases} 0, & t < 1, \\ 1/2, & t = 1, \\ 1, & t > 1. \end{cases}$$

For Gaussian X or, more generally, for X in the domain of attraction of a stable law of exponent $\alpha \in (0, 2]$, there are precise expressions for $\mathbf{E}(\mu_n(B))$ for any Borel set $B \subset \mathbb{C}$; see [20] and [12].

5.2. Independent (not necessarily equidistributed) coefficients. Let us consider now a sequence $(X_n)_{n \geq 0}$ of mutually independent random variables in a certain probability space (Ω, \mathbf{P}) ; no assumption now on a common distribution. As above, we let

$$f_\omega(z) = \sum_{n=0}^{\infty} X_n(\omega) z^n.$$

For $n \geq 0$ and $\gamma \in (0, 1)$, we denote $A_n(\gamma) = \max_{(1-\gamma)n \leq k \leq n} |X_k|$.

After reviewing the discussion above of the iid case, it is easy to come out with natural and simple conditions on the sequence of independent variables $(X_n)_{n \geq 0}$ which are enough to guarantee the conclusions of Lemmas 5.1 and 5.2.

A) If for some $\varepsilon > 0$, one has

$$(5.1) \quad \sup_{n \geq 0} \mathbf{E}((\ln^+ |X_n|)^{1+\varepsilon}) < +\infty,$$

then for $\alpha > 0$ and $n \geq 0$, Markov's inequality gives us that

$$\mathbf{P}(|X_n| \geq e^{\alpha n}) \leq \frac{\mathbf{E}((\ln^+ |X_n|)^{1+\varepsilon})}{\alpha^{1+\varepsilon} n^{1+\varepsilon}}.$$

Therefore

$$\sum_{n=0}^{\infty} \mathbf{P}(|X_n| \geq e^{\alpha n}) < +\infty, \quad \text{for each } \alpha > 0,$$

and, consequently, see the proof of Lemma 5.1), we conclude that

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|X_n|} \leq 1 \quad \text{almost surely.}$$

B) If for some $\delta > 0$, one has that

$$(5.2) \quad \inf_{n \geq 0} \mathbf{P}(|X_n| \geq \delta) > 0,$$

then (see the proof of Lemma 5.1)

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|X_n|} \geq 1 \quad \text{almost surely,}$$

and, besides, the proof of Lemma 5.2 carries over and gives that

$$\liminf_{n \rightarrow \infty} \sqrt[n]{A_n(\gamma)} \geq 1 \quad \text{almost surely.}$$

Therefore we have:

Theorem 5.7. *Under conditions (5.1) and (5.2) above, the random power series f_ω is almost surely a Szegő power series.*

Conditions analogous to (5.1) and (5.2) appear also in [18] to obtain a result like Theorem 5.7.

Remark 5.8. Conditions (5.1) and (5.2) are not as demanding than those appearing in [21]: no continuous densities or finite moments assumptions other than the log-moment above.

Under the assumptions of [21], Theorem 1, one actually has $\lim_{n \rightarrow \infty} \sqrt[n]{|X_n|} = 1$ almost surely, and thus almost surely Theorem 2.2 applies (no need to appeal to Theorem 2.7), and the sequence $(\mu_n)_{n \geq 0}$ of random probabilities converges almost surely to the uniform probability Λ on $\partial \mathbb{D}$.

Bernoulli trials. Let $(X_n)_{n \geq 0}$ be a sequence of completely independent Bernoulli variables with $p_n = \mathbf{P}(X_n = 1)$, for $n \geq 0$. Notice that condition (5.1) is trivially satisfied in this case.

Because of independence and the Borel–Cantelli lemmas, for the radius of convergence we have in this case that:

- a) if $\sum_{n=1}^{\infty} p_n = +\infty$, then almost surely the radius of convergence of f_ω is 1;

b) if $\sum_{n=1}^{\infty} p_n < +\infty$, then f_{ω} is almost surely a polynomial and its radius of convergence is $+\infty$.

As for belonging to \mathcal{S}'' , we have that if $\inf_{n \geq 0} p_n > 0$, then both conditions, (5.1) and (5.2), are satisfied and almost surely f_{ω} is in \mathcal{F} and also in \mathcal{S}'' .

Consider now the case where

$$(\dagger) \quad p_n = 1/n, \text{ for } n \geq 1.$$

In this case f_{ω} has radius of convergence 1, almost surely. Condition (5.2) is not satisfied and, in fact, as we shall presently verify, the index of f_{ω} is almost surely 1.

Fix $\gamma \in (0, 1)$ and let $(n_k)_{k \geq 1}$ such that $(1 - \gamma) > n_{k-1}/n_k \rightarrow (1 - \gamma)$.

Then $\mathbf{P}(A_{n_k}(\gamma) = 0) = \prod_{(1-\gamma)n_k \leq j \leq n_k} (1 - 1/j)$ and so, for k large enough,

$$\mathbf{P}(A_{n_k}(\gamma) = 0) \geq \exp\left(-2 \sum_{(1-\gamma)n_k \leq j \leq n_k} 1/j\right) \geq C(1 - \gamma)^2.$$

Since $(1 - \gamma) > n_{k-1}/n_k$, the events $\{A_{n_k}(\gamma)\}$ are independent. The lemma of Borel–Cantelli now gives that

$$\mathbf{P}(A_{n_k}(\gamma) = 0 \text{ infinitely often}) = 1,$$

and, consequently,

$$\mathbf{P}(L(\gamma) = 0) = 1, \quad \text{for each } \gamma \in (0, 1).$$

Therefore the index Γ is almost surely 1. In this case we are, almost surely, in the situation of Remark 4.1.

So, for probabilities p_n satisfying (\dagger) the random power series f_{ω} has almost surely radius of convergence 1, but almost surely f_{ω} is not a Szegő power series.

6. LIMITS OF ZERO COUNTING MEASURES

It is natural to ask what are the possible (weak) limits of the sequence of probabilities ρ_n associated to a given $f \in \mathcal{F}$. Let us denote by \mathcal{L}_f the collection of those limits points; the elements of \mathcal{L}_f are probability measures on $[0, +\infty]$.

If the index Γ of f is 0, then, by Theorem 2.6, the only such limit is δ_1 , i.e. $\mathcal{L}_f = \{\delta_1\}$, and, conversely.

6.1. Power series with coefficients 0 or 1. If all the coefficients of the power series $f \in \mathcal{F}$ are 0 or 1 then we have the following complete description of \mathcal{L}_f :

Proposition 6.1. *If $f \in \mathcal{F}$ has index Γ , then*

$$\mathcal{L}_f = \{(1 - u)\delta_1 + u\delta_{\infty} : 0 \leq u \leq \Gamma\}.$$

Proof. Denote by \mathcal{M} the collection of indexes n such that $a_n = 1$. With no loss of generality we assume that $a_0 = 1$.

For $T > 1$ and $n \in \mathcal{M}$, equation (3.2) gives us

$$\ln(T) (1 - F_n(T)) \leq \frac{1}{2\pi} \int_0^{2\pi} \ln \sqrt[n]{|s_n(e^{i\vartheta})|^2} d\vartheta \leq \frac{2}{n} \ln(n+1),$$

and so, for any $T > 1$,

$$\lim_{\substack{n \in \mathcal{M}; \\ n \rightarrow \infty}} F_n(T) = 1.$$

Consequently, ρ_n tends to δ_1 as $n \in \mathcal{M}$ tends to ∞ .

For integer $n \geq 0$, let $m(n) = \max\{m \in \mathcal{M} : m \leq n\}$. Observe that

$$F_n \equiv \frac{m(n)}{n} F_{m(n)}, \quad \text{for } n \geq 0.$$

Notice also that each ρ_n has mass $1 - m(n)/n$ at $+\infty$.

Thus, for an increasing sequence $(n_k)_{k \geq 1}$ of indexes, the sequence ρ_{n_k} has limit, say, ρ if and only if the sequence $m(n_k)/n_k$ converges, say, to α ; in that case $\rho = \alpha\delta_1 + (1 - \alpha)\delta_{\infty}$.

Since the possible limits of sequences $m(n_k)/n_k$ cover exactly the interval $[1 - \Gamma, 1]$, the result follows. \square

The simple argument above is modeled upon part of the discussion of [4].

6.2. A universal power series. Let \mathcal{P} be the set of probability (Borel) measures in $[0, +\infty)$ and let \mathcal{P}_1 be the subset of \mathcal{P} of those probabilities supported in $[1, +\infty)$. We endow \mathcal{P} with the Lévy–Prokhorov metric (distance) D with respect to Euclidean distance in $[0, \infty)$; convergence with respect to this metric D and weak convergence coincide.

In this section we shall exhibit an example of a *single* power series $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{F}$ such that *every* probability measure in $[1, +\infty)$ is a limit of a subsequence of the probability measures $(\rho_n(f))_{n \geq 0}$ associated to the sequence of sections of f . The power series f is *universal* in the sense that the probabilities measures $\rho_n(f)$ are dense in \mathcal{P}_1 :

$$\text{clos}_D\{\rho_n(f) : n \geq 1\} = \mathcal{P}_1.$$

The countable collection $\mathcal{D} \subset \mathcal{P}_1$ of probabilities of the form $(1/m) \sum_{j=1}^m \delta_{q_j}$, where m is an integer $m \geq 1$ and $1 < q_1 < \dots < q_m$ are rational numbers, is dense in \mathcal{P}_1 . Let $(\varphi^{(k)})_{k \geq 1}$ be a sequence of probabilities which contains each of the probabilities in \mathcal{D} infinitely many times.

The power series $f \in \mathcal{F}$ will have the form

$$f(z) = 1 + \sum_{j=1}^{\infty} z^{N_j} Q_j(z).$$

The Q_j are polynomials with $Q_j(0) = 1$. Denote $P_k(z) = 1 + \sum_{j=1}^k z^{N_j} Q_j(z)$ and $d_k = \deg(P_k)$. The integers N_j grow so fast that $N_k > d_{k-1}$ for any $k \geq 1$. Thus $s_{d_k}(f) = P_k$, for $k \geq 1$.

The polynomials Q_j and the integers N_j will be defined iteratively so that

$$D(\rho_{d_k}(f), \varphi^{(k)}) \leq \frac{1}{k} \quad \text{for any } k \geq 1.$$

Before starting the actual construction we record a few preliminary lemmas. We shall need the following estimate of the distance D of two specific probabilities whose verification follows directly from the definition of D .

Lemma 6.2. *Let $\mu \in \mathcal{P}_1$ be given by $\mu = (1/m) \sum_{j=1}^m \delta_{r_j}$ where $1 < r_1 < \dots < r_m$.*

Let $\varepsilon > 0$ be such that the intervals $I_j(\varepsilon) := (r_j - \varepsilon, r_j + \varepsilon)$ are pairwise disjoint.

Let $\nu \in \mathcal{P}$ be given by

$$\frac{1}{mk + h} \left(\sum_{j=1}^m \sum_{l=1}^k \delta_{s_{j,l}} + \sum_{i=1}^h \delta_{t_i} \right)$$

where $s_{j,l} \in I_j(\varepsilon)$, for $1 \leq j \leq m, 1 \leq l \leq k$ and $t_i \geq 0$ for $1 \leq i \leq h$.

If $\frac{h}{mk} < \varepsilon$, then $D(\mu, \nu) < \varepsilon$.

For integer $M \geq 1$, we denote by \mathcal{U}_M the collection of the M -th roots of unity.

Lemma 6.3. *For integer $M \geq 1$ and radius $r > 0$, one has*

$$\left| 1 - \left(\frac{z}{r} \right)^M \right| \geq 3 - e, \quad \text{for any } z \text{ such that } \text{dist}(z, r\mathcal{U}_M) \geq r/M.$$

Proof. Let z be such that $|z - ru| = r/M$, where $u \in \mathcal{U}_M$. Write $z = r(u + w/M)$, with $|w| = 1$. We have that

$$\begin{aligned} \left| 1 - \left(\frac{z}{r} \right)^M \right| &= \left| 1 - \left(u + \frac{w}{M} \right)^M \right| = \left| \sum_{j=1}^M \binom{M}{j} \frac{w^j u^{M-j}}{M^j} \right| \geq 1 - \left| \sum_{j=2}^M \binom{M}{j} \frac{w^j u^{M-j}}{M^j} \right| \\ &\geq 1 - \sum_{j=2}^M \binom{M}{j} \frac{1}{M^j} = 1 - \left(\left(1 + \frac{1}{M} \right)^M - 2 \right) \geq 3 - e. \end{aligned}$$

Let now Ω be the domain $\Omega = \{z \in \mathbb{C} : \text{dist}(z, r\mathcal{U}_M) \geq (r/M)\}$, and let g be the function

$$g(z) = \frac{1}{1 - (z/r)^M}.$$

The function g is holomorphic and does not vanish in Ω . Since g is continuous up to the finite boundary of Ω , $|g|$ is bounded there by $1/(3-e)$ and $\lim_{z \rightarrow \infty} g(z) = 0$, the maximum principle shows that g is bounded everywhere in Ω by $1/(3-e)$. \square

Corollary 6.4. *Let $1 < r_1 < r_2 < \dots < r_m$, and let M be an integer $M \geq 1$. Then*

$$\left| \prod_{j=1}^m \left(1 - \left(\frac{z}{r_j} \right)^M \right) \right| \geq (3-e)^m, \quad \text{if } \text{dist}(z, r_j\mathcal{U}_M) \geq r_j/M \text{ for } 1 \leq j \leq m.$$

We are now ready to describe the iterative **construction of f** .

Start with $P_0 \equiv 1$. Suppose that we have completed $k-1$ steps in the construction of our power series and that so far we have a section, denoted P_{k-1} , which has degree d_{k-1} .

Write the target probability measure $\varphi^{(k)}$ as $\varphi^{(k)} = (1/m) \sum_{j=1}^m \delta_{r_j}$, where $1 < r_1 < \dots < r_m$. Denote

$$\tau = \min \left\{ \frac{r_j - r_{j-1}}{r_j + r_{j-1}}; 1 < j \leq m \right\}$$

and let $A = \max \{|P_{k-1}(z)|; |z| \leq 2r_m\}$.

For integers $N, M \geq 1$, to be determined shortly, we set

$$P_k(z) = P_{k-1}(z) + z^N \prod_{j=1}^m \left(1 - \left(\frac{z}{r_j} \right)^M \right).$$

To start with we require (\star_1) $N > d_{k-1}$. This gives that the coefficients up to index d_{k-1} of P_k and of P_{k-1} coincide. Observe that the degree d_k of P_k is $d_k = d_{k-1} + N + mM$.

Now, we may choose N large enough, depending only on m , so that if

$$P_k(z) - P_{k-1}(z) = \sum_{n=N}^{d_k} b_n z^n.$$

then $\sqrt[n]{b_n} \leq 1 + \frac{1}{k}$, for each $N \leq n \leq d_k$, no matter what $M \geq 1$ may be. For that purpose and since $r_j > 1$, it is enough to choose N so that

$$(\star_2) \quad \left(\frac{m}{\lceil m/2 \rceil} \right)^{1/N} \leq 1 + \frac{1}{k}.$$

Observe also that the coefficient of index N of P_k is 1. All this means that the final outcome of this iterative construction will be a power series in \mathcal{F} . We remark that this requirement upon N imposes no restriction on $M \geq 1$.

Next we study the distribution of the zeros of $P_k(z)$ with the aim of showing that the circular projection of the zero counting measure of P_k on the positive real axis is close to the given $\varphi^{(k)}$. We shall compare the location of the zeros of P_k and the location of the zeros of $z^N \prod_{j=1}^m (1 - (z/r_j)^M)$. The zeros of this last polynomial are $\bigcup_{j=1}^m r_j\mathcal{U}_M$ and $z = 0$, a total of N times.

We require next that M is so large that (b) $(1/M) \leq \tau$ and also that (bb) $r_1(1 - 1/M) > (1 + r_1)/2$. Because of (b), the disks $\{z \in \mathbb{C} : |z - r_j\eta| = r_j/M\}$ where $1 \leq j \leq m$ and $\eta \in \mathcal{U}_M$ are pairwise disjoint.

We apply Rouché's theorem in each of these disks. If z is such that $|z - r_j\eta| = r_j/M$ with $1 \leq j \leq m$ and $\eta \in \mathcal{U}_M$, then $|z| \leq (1 + 1/M)r_j \leq 2r_m$ and, therefore,

$$\left| P_k(z) - z^N \prod_{j=1}^m \left(1 - \left(\frac{z}{r_j} \right)^M \right) \right| = |P_{k-1}(z)| \leq A,$$

while, because of Corollary 6.4 and (bb),

$$\left| z^N \prod_{j=1}^m \left(1 - \left(\frac{z}{r_j} \right)^M \right) \right| \geq \left(\frac{r_1 + 1}{2} \right)^N (3 - e)^m.$$

Therefore, if N is such that $(\star_3) \quad ((r_1 + 1)/2)^N (3 - e)^m > A$, the polynomial $P_k(z)$ has one zero in each of the disks $\{z \in \mathbb{C} : |z - r_j \eta| = r_j/M\}$. This occurs no matter what the value of M is, as long as (b) and (bb) are satisfied.

Fix N satisfying all the conditions (\star_i) above.

Finally, choose M satisfying, besides (b) and (bb) above, that $r_m/M \leq 1/k$ and that $N + d_{k-1} < mM/k$. Lemma 6.2 now gives us that

$$D(\rho, \varphi^{(k)}) \leq \frac{1}{k}$$

where $\rho = \frac{1}{d_k} \sum_{w \in \mathcal{Z}_{P_k}} \delta_{|w|}$.

We iterate this construction. The final outcome is a power series $f \in \mathcal{F}$ whose associated probabilities $\rho_n(f)$ satisfy, as desired,

$$D(\rho_{d_k}, \varphi^{(k)}) \leq \frac{1}{k}, \quad \text{for each } k \geq 1.$$

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